

BROWNSON

On a Locus Determined by
Concurrent Tangents
To a Quartic Curve

Mathematics

B. A.

1908

UNIVERSITY OF ILLINOIS
LIBRARY

Class

1908

Book

B 82

Volume

My 08 15M



ON A LOCUS DETERMINED BY CONCURRENT TANGENTS TO A QUARTIC CURVE

BY

WINNINA ELLA BROWNSON.

-O-

THESIS FOR THE DEGREE OF BACHELOR OF ARTS IN MATHEMATICS

IN THE

COLLEGE OF LITERATURE AND ARTS

OF THE

UNIVERSITY OF ILLINOIS.

-O-

June 1, 1908.

1908
B82

RECEIVED
LIBRARY OF THE
BUREAU OF THE
NAVY

UNIVERSITY OF ILLINOIS

May 28 1908

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Miss Winnina Ella Brownson

ENTITLED On a Locus Determined by Concurrent Tangents to a
Quartic Curve

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Bachelor of Arts in Mathematics

Charles Kesselhel Sisam

Instructor in Charge.

APPROVED:

E. J. Brownson

HEAD OF DEPARTMENT OF Mathematics

114563

Digitized by the Internet Archive
in 2013

On a Locus Determined by Concurrent Tangents to a Quartic Curve.

Let there be given a plane curve of the fourth degree. If an arbitrary line be drawn cutting the curve, and if the tangents be drawn at the points of intersection of the line with the quartic, in general no three of these tangents concur. However there exists a certain line locus, in the plane, such that the tangents to the quartic, at the three points of intersection of each line of the locus with the quartic meet in a point.

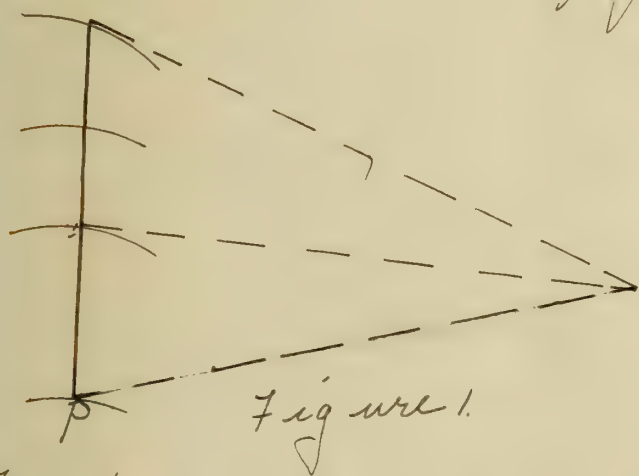
The problem which will now be considered is the determination of the class of this locus for various configurations of the fundamental quartic. The class of the locus in question is by definition equal to the number of lines of the locus which pass through an arbitrary point P on the given quartic curve.



These lines through P on the quartic are of two distinct types.

Type I.

The lines of the locus at P , which are of type I, are the lines drawn through P such that the tangents at two of the remaining intersections of each of the lines with the curve meet on the tangent drawn at P . Such a line is shown in figure 1.



Type II

The lines of the locus which are of type II at P are those such that the tangents at all three of the remaining intersections of each of the lines with the curve meet in one point. [Not on tangent drawn at P] Such a line is shown in figure 2.

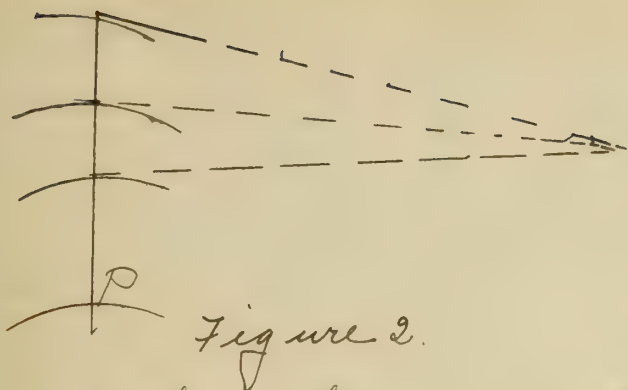


Figure 2.

The class of the locus is obtained by determining the number of lines of each of these types separately, then adding the two results. This gives the Class of the locus required.

Type I.

It is known that the number of lines of this type which pass through the arbitrary point P on the curve is in general equal to $[m-2][n-2]-p$ (note 1). If however the quartic has a Tac-

(note 1) Notation used.

m = The order of the given quartic.

n = The class of the given quartic.

p = genus or deficiency of curve.

S = Number of lines of the locus of type I.

R = Number of lines of the locus of type II.

node or Rhamphoid cusp this number must⁴
be reduced by one. If it has a Tacnode
cusp or an Acnode the number must
be reduced by two.

Type II.

The number of lines of the second
type will be determined analytically.
Let P be an arbitrary point on the
quartic and let its homogeneous coor-
dinates be $(0, 0, 1)$. It is necessary to
find the number of lines through $(0, 0, 1)$
such that the tangents to the quartic
at the three remaining points of
intersection of the line with the
quartic all meet in one point.

Let $f(x, y, z) = 0$ be the homo-
geneous equation of the quartic. If
we put $y = kx$ in this equation
and also in equation $x'f'_x + y'f'_y + z'f'_z = 0$
of the tangent, we obtain two
cubic equations, homogeneous in
 x and z , whose roots must all
be equal, when x', y', z' are the

coordinates of the point at which the three tangents all concur. The coefficients of x^3 , x^2z , xz^2 and z^3 of one equation are then proportional to those of the other. Let λ be the factor of proportionality.

Thus four linear homogeneous equations in x' , y' , z' and λ are obtained.

Eliminating the x' , y' , z' and λ we obtain an equation in " b " the number of whose roots is equal to the number of lines of the second type through $(0, 0, 1)$.

It will some times be convenient to take the point $(0, 0, 1)$ to be not a single or a simple point of the given quartic but a node or a cusp.

If $(0, 0, 1)$ is a node the lines of the locus through $(0, 0, 1)$ are the following: (1) The nodal tangents, (2) the remaining tangents from the quartic to the node and (3) the lines through the node such that the tangents to the quartic at the two remaining intersections of each of the lines with the quartic meet on one or the other

6

of the tangent at the node.

If $(0, 0, 1)$ is a cusp the lines of the locus through $(0, 0, 1)$ are the following; (1) the cuspidal tangent, (2) the remaining tangents to the quartic from the cusp; (3) the lines through the cusp such that the tangents to the quartic at the two remaining intersections of each of these lines with the quartic meet on the cuspidal tangent. Each of the lines of the third kind is tangent to its own locus at the cusp, and is therefore to be counted twice.

No Singularities.

7

The equation in homogeneous coordinates of the quartic with no singularities, which passes through $(0, 0, 1)$ is:

$$Ax^4 + x^3(By + Cz) + x^2(Dy^2 + Eyz + Fz^2) + x(Gy^3 + Hy^2z + Iyz^2 + Jz^3) + Kx^4 + My^3z + Ny^2z^2 + Oz^3 = 0$$



The number of lines of type I is equal to $[m-2][n-2] - p$.

$$(4-2)(12-2) - 3 = 17 \therefore S = 17.$$

To determine the number of lines of type II, form the equation of the tangent at (x, y, z) which is:

$$[4Ax^3 + 3Bx^2y + 3Cx^2z + 2Dxy^2 + 2Exyz + 2Fxz^2 + Gy^3 + Hy^2z + Iyz^2 + Jz^3]x' + [3x^3 + 2Dx^2y + Exy^2 + 3Gxy^2 + 2Hxy^2 + Iyz^2 + 4Kyz^3 + 3My^2z^2 + Nx^2z + 2Oxz^2 + 2Fyz^2 + 4xy^2 + 2Dxyz + 3Jxz^2 + My^3 + 2Ny^2z + 3Oyz^2]z' = 0$$

8

Let $y = kx$ in the equation of the tangent and in the equation of the Curve. Collecting the coefficients of x^3 , x^2z , xz^2 and z^3 in each equation and putting each coefficient in the former equal to $-\lambda$ times the corresponding coefficient in the latter we obtain:

$$[4A + 3Bk + 3Dk^2 + Gk^3]x' + [5 + 2Dk + 3Gk^2 + 4Kk^3]y' + [C + Ek + Hk^2 + Mk^3]z' + [F + Bk + Dk^2 + Gk^3 + Kk^4]\lambda = 0$$

$$[3C + 2Ek + Hk^2]x' + [E + 2Hk + 3Mk^2]y' + [2F + 2Bk + 2Gk^2]z' + [C + Ek + Hk^2 + Mk^3]\lambda = 0$$

$$[2F + Dk]x' + [5 + 2Gk]y' + [3J + 3Ok]z' + [F + Dk + Mk^2]\lambda = 0$$

$$Jx' + Oy' + [J + Ok]\lambda = 0$$

Eliminating x' , y' , z' , λ from these four linear homogeneous equations we obtain the following determinant:

$$4A + 3Bk + 2Dk^2 + 9k^3$$

$$B + 2Pk + 3bk^2 + 4Kk^3$$

$$C + Ek + Kk^2 + Mk^3,$$

$$A + Bk + Ak^2 + 9k^3 + Kk^4$$

$$3C + 2Ek + Kk^2$$

$$E + 2Kk + 3Mk^2$$

$$2F + 2Lk + 2Nk^2$$

$$C + Ek + Kk^2 + Mk^3$$

$$2F + Lk$$

$$L + 2Nk$$

$$3J + 3Ok$$

$$F + Lk + Kk^2$$

$$J$$

$$O$$

$$O$$

$$J + Ok$$

= 0

This equation is of the seventh degree in " k ".
Hence there are seven lines of type II
through $(0, 0, 1)$

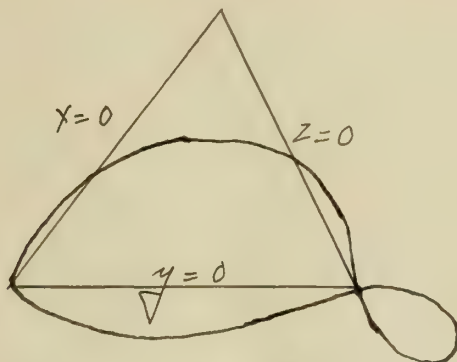
$$S \equiv 17 \quad R \equiv 7$$

$$S + R = 17 + 7 = 24$$

Therefore there are twenty-four lines so
the class of the required locus curve is
twenty-four.

One Double Point

The equation in homogeneous co-ordinates of the quartic, with one double point at $(1, 0, 0)$, which passes through $(0, 0, 1)$ and $z=0 = \tan$ is;

$$x^2(Eyz + Fz^2) + x(Gy^3 + Hy^2z + Iyz^2 + Jz^3) + Ry^4 + My^3z + Ny^2z^2 + Oyz^3 = 0$$


The number of lines of type I is equal to $(m-2)[n-2] - p$.

$$(4-2)(10-2) - 2 = 14 \therefore S = 14$$

To determine the number of lines of type II, form the equation of the tangent at (x, y, z) which is:

$$\begin{aligned} & [2Exyz + 2Fxy^2 + Gy^3 + Hy^2z + Iyz^2 + Jz^3]x' + \\ & [Ex^2z + 3Gxy^2 + 2Hxy^2 + Ix^2z + 4Ky^3 + 3My^2z + 2Ny^2z^2 + Oz^3]y' + \\ & [Ex^2y + 2Fxy^2 + Hxy^2 + 2Ixy^2 + 3Jx^2z + My^3 + 2Ny^2z + 3Oyz^2]z' = 0 \end{aligned}$$

Let $y = kx$ in the equation of the tangent and in the equation of the curve. Collecting the coefficients of.

$x^3, x^2, x, 1$ in each equation and putting each coefficient in the former equation equal to $-\lambda$ times the corresponding coefficient in the latter we obtain:

$$gb^2x' + [3gk + 4kb^2]y' + [E + Hk + mb^2]z' + [gb^2 + kb^3]\lambda = 0$$

$$[2Ekx' + Hk^2x' + [E + 2Hk + 3mb^2]y' + [2F + 2gk + 3nb^2]z' + [Ek + Hk^2 + mk^3]\lambda = 0$$

$$[2F + gk]x' + [2 + 2nk]y' + [3]z' + 3Ok]z' + [F + gk + nb^2]\lambda = 0$$

$$Jx' + 0y' + [J + Ok]\lambda = 0$$

Eliminating x', y', z' and λ from these four linear homogeneous equations we obtain the following determinant:

gb^2	$3gk + 4kb^2$	$E + Hk + mb^2$	$gb^2 + kb^3$
$2Ek + kb^2$	$E + 2Hk + 3mb^2$	$2F + 2gk + 2nb^2$	$Ek + Hk^2 + mk^3$
$2F + gk$	$2 + 2nk$	$3J + 3Ok$	$F + gk + nb^2$
J	0	0	$J + Ok$

This equation is of the sixth degree in " k ". Hence there are six lines of type II through $(0, 0, 1)$

$$S = 14 \quad R = 6.$$

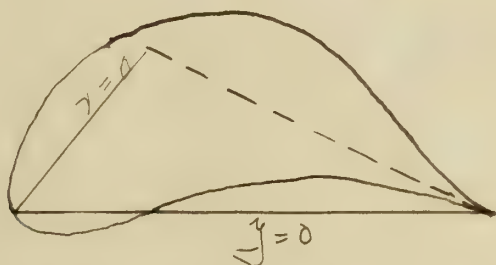
$$S + R = 14 + 6 = 20$$

Therefore the class of the required locus curve is twenty.

One Cusp.

The equation in homogeneous coordinates of the quartic curve with one cusp at $(1, 0, 0)$ is:

$$D^2x^2z^2 + Gxy^3 + Hy^2xz + Ixy^2z^2 + Jxz^3 + Ky^4 + Mxz^3 + Ny^2z^2 + Oz^3 = 0$$



The number of lines of type I is equal to $[m-2][n-2]-p \equiv S$.

$$[4-2][9-2]-2 = 12 \therefore S = 12$$

To determine the number of lines of type II, form the equation of the tangent at (x, y, z) which is:

$$[2D^2xz^2 + Gy^3 + Hy^2z + Ixy^2z^2 + Jz^3]x' + [3Gxy^2 + 2Hxz^2 + 2Ixy^2z + 4Ky^3 + 3Mxz^2 + 2Ny^2z^2 + Oz^3]y' + [2D^2x^2z + Hxy^2 + 2Ixyz + 3Jxz^2 + My^3 + 2Ny^2z + 3Oxz^2]z' = 0$$

Let $y = kx$ in the equation of the tangent and in the equation of the curve. Collecting the coefficients of x^3 ,

x^2, xz and z^3 in each equation and putting each coefficient in the former equation equal to $-x$ times the corresponding coefficient in the latter we obtain:

$$Gkx' + [3G + 4Kk]y' + [H + Mk]z' + [Gk + Kk^2]x = 0$$

$$Kk^2x' + [2Kk + 3Mk^2]y' + [2D^2 + 2Gk + 2Nk^2]z' + [Hk^2 + Mk^3]x = 0$$

$$[2D^2 + Gk]x' + [D + 2Nk]y' + [3J + 3Ok]z' + [D^2 + Gk + Mk^2]x = 0$$

$$Jx' + 0y' + [J + Ok]x = 0.$$

Eliminating x', y', z' and x from these four linear homogeneous equations, we obtain the following determinant.

Gk	$3G + 4Kk$	$H + Mk$	$Gk + Kk^2$	$= 0$
Kk^2	$2Kk + 3Mk^2$	$2D^2 + Gk + 2Nk^2$	$Hk^2 + Mk^3$	
$2D^2 + Gk$	$D + 2Nk$	$3J + 3Ok$	$D^2 + Gk + Mk^2$	
J	0	0	$J + Ok$	

This equation is of the fifth degree in "b". Hence there are five lines of type II through (0, 0, 1).

$$S = 12 \quad R = 5$$

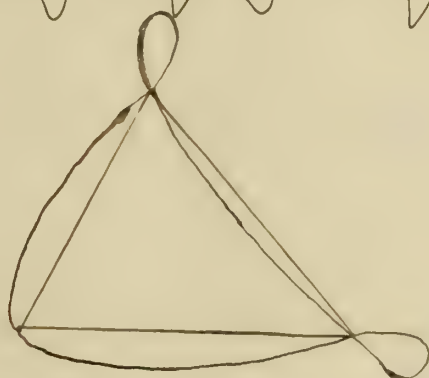
$$S + R = 12 + 5 = 17.$$

Therefore the class of the required locus curve is seventeen.

Two Double Points.

The equation in homogeneous coordinates of the quartic curve with two double points one at $(1, 0, 0)$ the other at $(0, 1, 0)$ and passing through the point $(0, 0, 1)$ is:

$$2Dx^2y^2 + Ex^2yz + Fx^2z^2 + Hy^2xz + Ixyz^2 + Jx^3 + Ky^3z^2 + Oz^3 = 0$$



The number of lines of type I is equal to $(m-2)(n-2) - p$.

$$[4-2][6-2] - 1 = 11 \therefore S \equiv 11$$

To determine the number of lines of type II, form the equation of the tangent at (x, y, z) , which is:

$$\begin{aligned} & [2Dxy^2 + 2Exyz + 2Fxz^2 + Hy^2z + Iyz^2 + Jz^3]x' \\ & + [2Dx^2y + Ex^2z + 3Hxyz + Ix^2z^2 + 2Kyz^2 + Oz^3]y' \\ & + [2Dx^2y + 2Ex^2z + Hxy^2 + 2Ixyz + 3Jxz^2 + 2Kyz^2 + 3Oz^3]z' = 0 \end{aligned}$$

Let $y = kx$ in the equation of the tangent and in the equation of the curve. Collecting the coefficients of



x^3, x^2, x and z^3 in each equation and putting each coefficient in the former equation equal to $-\lambda$ times the corresponding coefficient in the latter we obtain:

$$(2Dk)x' + (2D)y' + (C + Hk)z' + (Dk)\lambda = 0$$

$$[2Ek + Hk^2]x' + [C + 2Hk]y' + (2F + 2Dk + 2Hk^2)z' + [Ek + Hk^2]\lambda = 0$$

$$[2F + Dk]x' + (2 + 2Hk)y' + (3J + 3Dk)z' + (F + Dk + Hk^2)\lambda = 0$$

$$Jx' + 0y' + (J + Dk)\lambda = 0$$

Eliminating x', y', z' and λ from these four linear homogeneous equations we obtain the following determinant:

$2Dk$	$2D$	$C + Hk$	Dk	
$2Ek + Hk^2$	$C + 2Hk$	$2F + 2Dk + 2Hk^2$	$Ek + Hk^2$	
$2F + Dk$	$J + 2Hk$	$3J + 3Dk$	$F + Dk + Hk^2$	$= 0$
J	0	0	$J + Dk$	



This equation is of fifth degree in k . Hence there are five lines of type II through $(0,0,1)$

$$S = 11 \quad R = 5$$

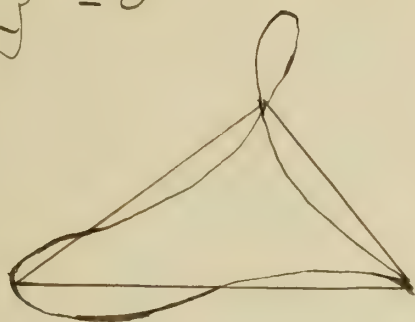
$$S + R = 11 + 5 = 16$$

Therefore the class of the required locus curve is sixteen.

One double point and one Cusp.

The equation of the quartic in homogeneous coordinates having a double point at $(0, 1, 0)$ and a cusp at the point $(1, 0, 0)$ is:

$$D^2 x^2 y^2 + 2F D x^2 y z + F^2 x^2 z^2 + H x y^2 z + I x y z^2 + J x z^3 + K y^2 z^2 + O y z^3 = 0$$



The number of lines of type I is equal to $[m-2][n-2]-p=5$
 $[4-2][7-2]-1 = 9 \therefore S=9$

To determine the number of lines of type II. Form the tangent equation at (x, y, z) which is:

$$[2 D^2 x y^2 + 4 F D x y z + 2 F^2 x z^2 + 4 y^2 z + I y z^2 + J z^3] x' + [2 D^2 x^2 y + 2 F D x^2 z + 2 H x y z + I x z^2 + 2 K y z^2 + O z^3] y' + [2 F D x^2 y + 2 F^2 x^2 z + H x y^2 + 2 I x y z + 3 J x z^2 + 2 K y^2 z + 3 O y z^2] z' = 0$$

Let $kx=y$ in the equation of the

tangent and in the equation of the curve. Collecting the coefficients of x^3, x^2, xz, z^3 in each equation and putting each coefficient in the former equation $\rightarrow$ times the corresponding coefficient in the latter we obtain;

$$\begin{aligned} & [2D^2k]x' + 2D^2y' + [2FD + 14k]z' + [D^2k]\lambda = 0 \\ & [4FDk + 14k^2]x' + [2FD + 24k]y' + [2F + 2Dk + 2k^2]z' + [2FDk + 14k^2]\lambda = 0 \\ & [2F + Dk]x' + [D + 2k]y' + [3]z' + [2Dk]\lambda = 0 \\ & [F]x' + 0y' + [F + Dk]\lambda = 0 \end{aligned}$$

Eliminating x', y', z', λ from these four linear homogeneous equations we obtain the following determinant;

$2D^2k$	$2D^2$	$2FD + 14k$	D^2k
$4FDk + 14k^2$	$2FD + 24k$	$2F^2 + 2Dk + 2k^2$	$2FDk + 14k^2$
$2F^2 + Dk$	$D + 2k$	$3F + 3Dk$	$F^2 + Dk + k^2$
F	0	0	$F + Dk$

This equation is of fourth degree in k . Hence there are four lines of type II through $(0, 0, 1)$

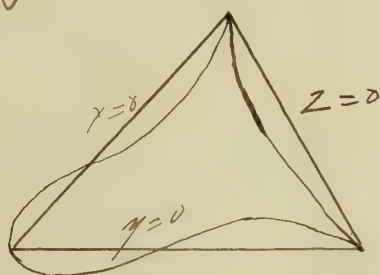
$$L \equiv 9 \quad R \equiv 4$$
$$L + R = 9 + 4 = 13$$

Therefore the class of the required locus curve is thirteen.

Two Cusps.

The equation in homogeneous coordinates of the quartic with two cusps one at $(1, 0, 0)$ and the other at $(0, 1, 0)$ is:

$$D^2 x^2 y^2 + 2FD x^2 y z + F^2 x^2 z^2 + 2ND x y^2 z + D x y z^2 + D x z^3 + N^2 y^2 z^2 + D y z^3 = 0$$



The number of lines of type I is equal to $[m-2][n-2]-p$

$$[4-2][6-2]-1 = 7 \quad \therefore S=7.$$

To determine the number of lines of type II, form the equation of the tangent at (x, y, z) which is:

$$\begin{aligned} & [2D^2 x y^2 + 4FD x y z + 2F^2 x z^2 + 2ND y^2 z + D y z^2 + D^3] x' \\ & + [2D^2 x^2 y + 2FD x^2 z + 4ND x y z + D x z^2 + 2N^2 y z^2 + D^3] y' \\ & + [2FD x^2 y + 2F^2 x^2 z + 2ND x y^2 + 2D x y z + 3D^2 x z^2 + 2N^2 y^2 z + 3D y z^2] z' = 0 \end{aligned}$$

Let $y = kx$ in both equations, that of the curve and that of the tangent.

Collecting the coefficients of x^3, x^2, x and z^3 in each equation and putting each coefficient in the former equation equal to $-\lambda$ times the corresponding coefficient in the latter we obtain:

$$[2D^2k]x' + [2D^2]y' + [2FD + 2HDb]z' + D^2k\lambda = 0$$

$$[2FDk + HDb^2]x' + [FD + 2HDb]y' + [F^2 + Dk + H^2b^2]z' + [FDk + HDb^2]\lambda = 0$$

$$[2F^2 + Dk]x' + [D + 2H^2b]y' + [3] + 3Db]z' + [F^2 + Dk + H^2b^2]\lambda = 0$$

$$[F]x' + [0]y' + [F + Db]\lambda = 0$$

Eliminating x', y', z' and λ from these four linear homogeneous equations we obtain the following determinant;

$2D^2k$	$2D^2$	$2FD + 2HDb$	D^2k
$2FDk + HDb^2$	$FD + 2HDb$	$F^2 + Dk + H^2b^2$	$FDk + HDb^2$
$2F^2 + Dk$	$D + 2H^2b$	$3 + 3Db$	$F^2 + Dk + H^2b^2$
F	0	0	$F + Db$

24

This equation is of the third degree in k . Hence there are three lines of type II through $(0, 0, 1)$.

$$S = 7 \quad R = 3.$$

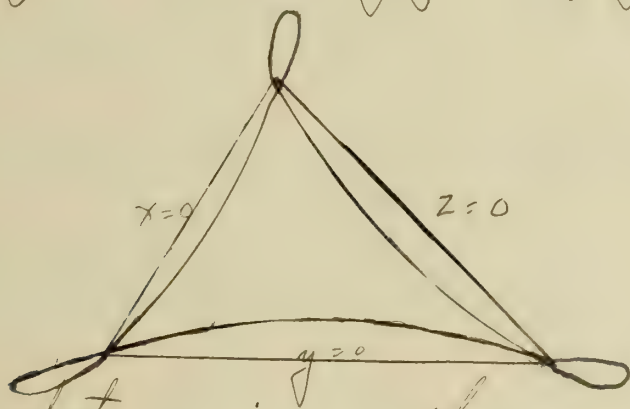
$$S + R = 7 + 3 = 10$$

Therefore the class of the required locus curve is ten.

Three Nodes.

The equation in homogeneous co-ordinates of the quartic curve with three nodes one at $(0, 0, 1)$ another at the point $(0, 1, 0)$, the third at the point $(1, 0, 0)$ is;

$$x^2[Fz^2 + Eyz + Dy^2] + x[Dyz^2 + Hy^2z] + Hy^2z^2 = 0$$



To determine the number of lines through $(0, 0, 1)$ which join the points of tangency[#] of two tangents which meet on one of the nodal tangents at $(0, 0, 1)$, form the equation of the tangent at (x, y, z) which is;

$$\begin{aligned} & [2Fxz^2 + 2Exyz + 2Dxy^2 + 2yz^2 + Hy^2z] x' \\ & + [Ex^2 + 2Dx^2y + 2xz^2 + 2Hxyz + 2Hy^2z] y' \\ & + [2Fx^2 + Ex^2y + 2Dxyz + Hxy^2 + 2Hy^2z] z' = 0 \end{aligned}$$

See page 5.

Let $y = kx$ in the equation of the tangent and in the equation of the curve. Collecting the coefficients of x^3, x^2, x and z^3 in each equation and putting each coefficient in the former equal to $-\lambda$ times the corresponding coefficient in the latter we have

$$2Dk x' + 2Dy' + [E + Hk]z' + Dk\lambda = 0$$

$$[2Ek + Hk^2]x' + [E + 2Hk]y' + [2F + 2Hk + 2Hk^2]z' + [Ek + Hk^2]\lambda = 0$$

$$[2F + Hk]x' + [D + 2Hk]y' + [F + Hk + Hk^2]\lambda = 0.$$

Let $D = F + H$ then $x' + y' = 0$ is one of the tangents at the node. Eliminating x', y' and z' and λ from the above three equations and the equation $x' + y' = 0$ of the tangent we obtain the determinant:

$2Dk$	$2D$	$E + Hk$	Dk	
$2Ek + Hk^2$	$E + 2Hk$	$2F + 2Hk + 2Hk^2$	$Ek + Hk^2$	$= 0$
$2F + Hk$	$D + 2Hk$	0	$F + Hk + Hk^2$	
1	1	0	0	

This equation is of the fifth degree in " k " but as $(k+1)$ is a factor it reduces to one of the fourth degree. Hence there are four lines through $(0, 0, 1)$ which are determined by the tangent $x' + y' = 0$

The same number are determined by the second nodal tangent.

The two nodal tangents at $(0, 0, 1)$ are lines of the locus as are also the $(n-4) = 2$ other tangents to the given quartic which can be drawn from $(0, 0, 1)$

The sum of the number of lines of the three different kinds[#] is;

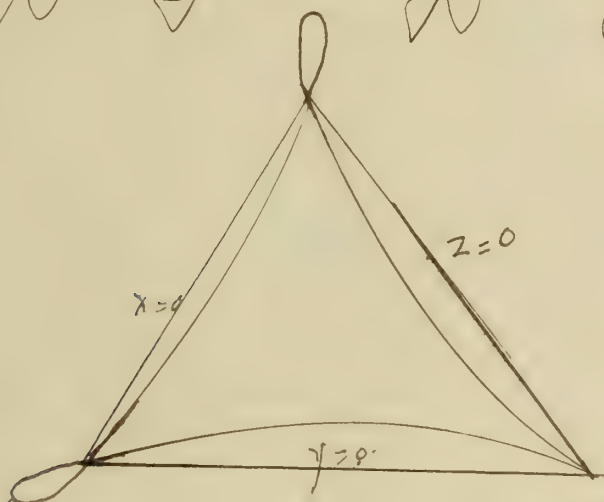
$$8 + 2 + 2 = 12.$$

Hence the class of the required locus is twelve.

Two Nodes ^{2nd} and One Cusp.

The equation in homogeneous coordinates of the quartic curve with two nodes and a cusp. The nodes at $(0, 1, 0)$ and $(0, 0, 1)$, the cusp at the point $(1, 0, 0)$ is;

$$x^2[Fz^2 + 2FDyz + D^2y^2] + x[Dyz^2 + Hy^2z] + Ny^2z^2 = 0$$



To determine the number of lines through $(0, 0, 1)$ which join the points of tangency of two tangents which meet on one of the nodal tangents[#] at $(0, 0, 1)$, form the equation of the tangent at (x, y, z) which is;

$$[2F^2xz^2 + 4FDxy^2 + 2D^2yz^2 + Dy^2z^2 + Hy^2z^2]x' + [2FDx^2z + 2D^2x^2y + Dx^2y^2 + 2Hxy^2z + 2Ny^2z^2]y' + [2F^2x^2z + 2FDx^2y + 2D^2xy^2 + Hy^2z^2 + 2Ny^2z^2]z' = 0$$

Let $y = kx$ be the equation of the tangent and in the equation of the curve.

see page 5.

Collecting the coefficients of x^3 , x^2y , x^2z^2 and z^3 in each equation and putting each coefficient in the former equal to $-\lambda$ times the corresponding coefficient in the latter we have;

$$\begin{aligned} [2D^3kx' + 2D^2y' + [2FD + Hk]z' + D^2k\lambda] &= 0 \\ [4FDk + Hk^2]x' + [2FD + 2Hk]y' + [2F^2 + 2Dk]z' + [2FDk + Hk^2]\lambda &= 0 \\ [2F^2 + Dk]x' + [D + 2Hk]y' + [F^2 + Dk + Hk^2]z' &= 0 \end{aligned}$$

Let $D = F + H$ then $x' + y' = 0$ is one of the tangents at the node. Eliminating x' , y' , z' and λ from the above three equations and the equation $x' + y' = 0$ of the tangent we obtain the determinant;

$2D^3k$	$2D^2$	$2FD + Hk$	D^2k	
$4FDk + Hk^2$	$2FD + 2Hk$	$2F^2 + 2Dk + 2Hk^2$	$2FDk + Hk^2$	
$2F^2 + Dk$	$D + 2Hk$	0	$F^2 + Dk + Hk^2$	
1	1	0	0	$= 0$

This equation is of the fourth degree in "b" but as $(b+1)$ is a factor, it reduces to one of the third degree. Hence there are three lines through $(0,0,1)$, which are determined by the tangent $x' + y' = 0$.

The same number are determined by the second nodal tangent.

The two nodal tangents at $(0,0,1)$ are lines of the locus & are also the $n-4=1$ other tangent to the given quartic which can be drawn from $(0,0,1)$

The sum of the number of lines of the three different kind^s is.

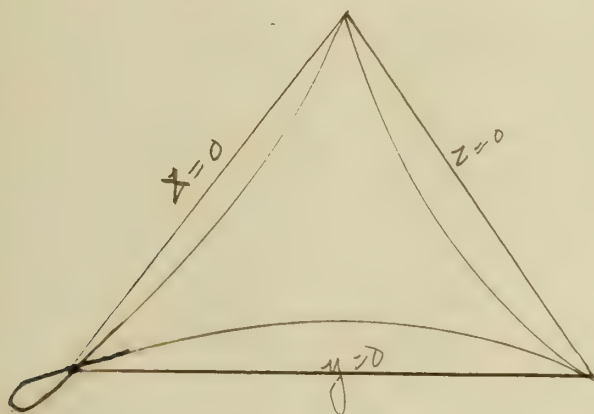
$$6 + 2 + 1 = 9.$$

Hence the class of the required locus is nine.

Two Cusps and One Node.

The equation in homogeneous coordinates of the quartic curve with two cusps and one node, the cusps at the points $(1, 0, 0)$ and $(0, 1, 0)$, the node at $(0, 0, 1)$ is:

$$F^2 x^2 z^2 + 2FD x^2 yz + D^2 x^2 y^2 + I x y z^2 + 2ND x y^2 z + N^2 y^2 z^2 = 0$$

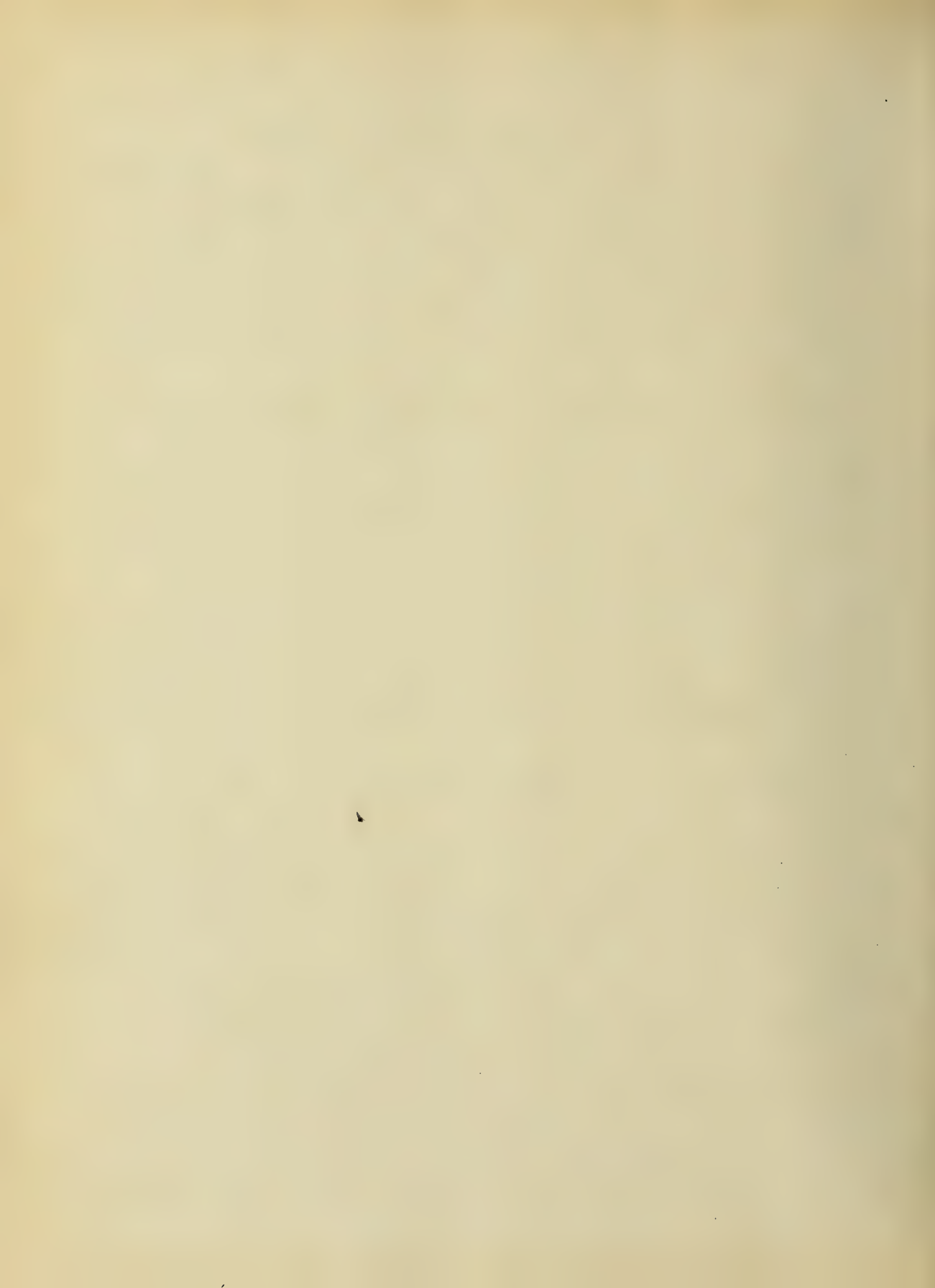


To determine the number of lines through $(0, 0, 1)$ which join the points of tangency[#] of two tangents which meet on one of the nodal tangents at $(0, 0, 1)$, form the equation of the tangent at (x, y, z) which is:

$$[2F^2 x z^2 + 4FD x y z + 2D^2 x y^2 + I y z^2 + 2ND y^2 z] x' + [2FD x^2 z + 2D^2 x^2 y + I x z^2 + 4ND x y z + 2N^2 y z^2] y' + [2F^2 x^2 z + 2FD x^2 y + 2I x y z + 2ND x y^2 + 2N^2 y^2 z] z' = 0$$

Let $y = kx$ in the equation of the

See page 5.



tangent and in the equation of the ³² curve. Collecting the coefficients of x^3 , x^2z , xz^2 and z^3 in each equation and putting each coefficient in the former equal to $-\lambda$ times the corresponding coefficient in the latter we have,

$$2D^2kx' + 2D^2y' + [2FD + 2HDbk]z' + D^2k\lambda = 0$$

$$[4FDk + 2HDbk^2]x' + [2FD + 2HDbk]y' + [2F^2 + 2Hk]z' + [2FDk + 2HDbk^2]\lambda = 0$$

$$[2F^2 + Hk]x' + [2 + 2H^2k]y' + [F^2 + Hk + H^2k^2]\lambda = 0$$

Let $I = F + H$ the $x + y' = 0$ is one of the tangents at the node. Eliminating x' , y' , z' and λ from the above three equations and the equation $x + y' = 0$ of the tangent we obtain the determinant:

$2D^2k$	$2D^2$	$2FD + 2HDbk$	D^2k	
$4FDk + 2HDbk^2$	$2FD + 4HDbk$	$2[F^2 + Hk + H^2k^2]$	$2FDk + 2HDbk^2$	
$2F^2 + Hk$	$2 + 2H^2k$	0	$F^2 + Hk + H^2k^2$	
1	1	0	0	$= 0$

This equation is of the fourth degree in k but ^{since} as the coefficient of k^5 vanishes but as k and $(k+1)$ are both factors it reduces to one of the second degree.

Hence there are two lines through $(0, 0, 1)$ but the same number are determined by the second nodal tangent. ∇

The two nodal tangents at $(0, 0, 1)$ are lines of the locus, as are also the $n-4=0$ other tangents to the given quartic which can be drawn from $(0, 0, 1)$.

The sum of the number of lines of the three different kinds^{is} is;

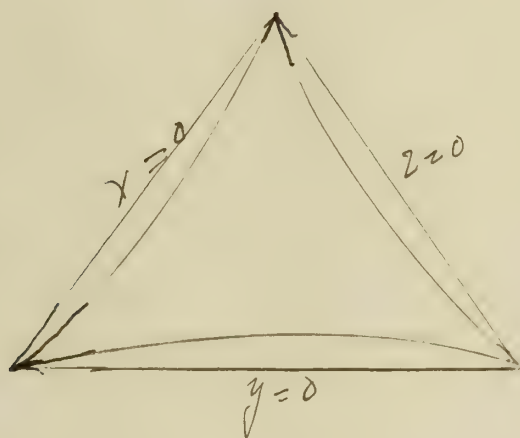
$$4 + 2 + 0 = 6.$$

Hence the class of the required locus is eight.

Three Cusps

The equation in homogeneous co-ordinates of the quartic curve with three cusps, one at $(0, 0, 1)$, another at $(0, 1, 0)$ the third at $(1, 0, 0)$ is:

$$F^2 x^2 z^2 + 2FD x^2 y z + D^2 x^2 y^2 + 2FN x y z^2 + 2ND x y^2 z + N^2 y^2 z^2 = 0$$



To determine the number of lines through $(0, 0, 1)$ which join the points of tangency* of two tangents which meet on the cuspidal tangent at $(0, 0, 1)$, form the equation of the tangent at (x, y, z) which is:

$$[2F^2 x z^2 + 4FD x y z + 2D^2 x y^2 + 2FN y z^2 + 2ND y^2 z + 2F^2 x^2 z + 2FD x^2 y + 4FN x y z + 2ND x y^2 + N^2 y^2 z^2] Z' = 0$$

Let $y = kx$ in the equation of the tangent and in the equation of the curve. Collecting [See page (5)] the coefficients of x^3 , $x^2 z$, $x z^2$ and z^3

xz^2 and z^3 in each equation and putting each coefficient in the former equal to $-\lambda$ times the corresponding coefficient in the latter we have;

$$2D^2kx' + 2D^2y' + (2FD + 2Nk)z' + D^2k\lambda = 0$$

$$[4FDk + 2NDk^2]x' + [2FD + 4NDk]y' + [2F^2 + 4FNk]z' + [2FDk + 2NDk^2]\lambda = 0$$

$$[2F^2 + 2FNk]x' + [2FN + 2N^2k]y' + [F^2 + 2FNk + N^2k^2]\lambda = 0$$

Let $F = H$ then $x' + y' = 0$ is the equation of the tangent at the cusp. Eliminating x', y', z' and λ from the three equations and the equation $x' + y' = 0$ of the tangent we obtain the determinant;

$2D^2k$	$2D^2$	$2FD + 2Nk$	D^2k
$4FDk + 2NDk^2$	$2F^2 + 4NDk$	$2(F^2 + 2FNk + N^2k^2)$	$2FDk + 2NDk^2$
$2F^2 + 2FNk$	$2FN + 2N^2k$	0	$F^2 + 2FNk + N^2k^2$
1	1	0	0

= 0

This equation is of the fourth degree in k since the coefficient of k^4 vanishes, but as k and $(k+1)^2$ are factors, it reduces to one of the first degree. Hence there is one line through $(0, 0, 1)$ but this line must be counted twice.

The cuspidal tangent at $(0, 0, 1)$ is a line of the locus.

Since $(n-3)=0$ no other lines of the locus, except the cuspidal tangent and the lines determined above pass through $(0, 0, 1)$.

The sum of the number of lines of the three different kinds[#] is:

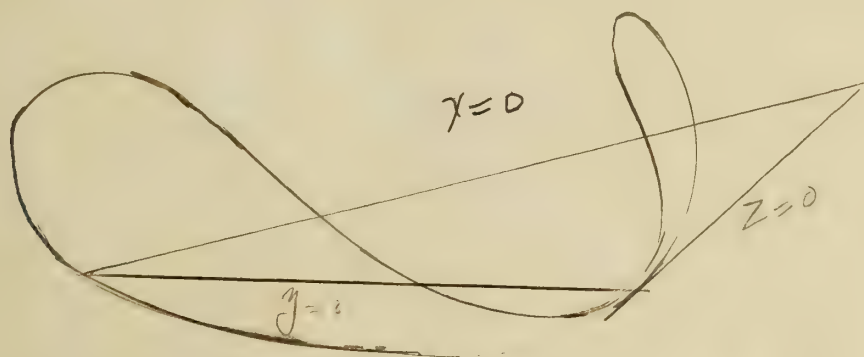
$$2 + 1 + 0 = 3$$

Hence the class of the required locus is three.

The Tacnode.

The equation in homogeneous coordinates of the quartic with a tacnode at $(1, 0, 0)$ and passing through the point $(0, 0, 1)$ is;

$$Lx^3 + Oxy^3 + F^2x^2z^2 + Ixy^2z^2 + Ny^3z^2 + Hxyz^3 + My^3z + Kxy^4 = 0$$



The number of lines of type I is equal to $(m-2)(n-2) - p - 1^* =$
 $[4-2][8-2] - 1 - 1 = 10$

To determine the number of lines of type II form the equation of the tangent at (x, y, z) which is;

$$L(3x^2 + 2Fxy^2 + 2y^2z^2 + Hy^3z^2)x' + (Ox^3 + Ixy^2z^2 + 2Ny^3z^2 + 2Hxyz^3 + 3My^4z^2 + 4Kxy^3z^2)y' + (3Lx^2z^2 + 3Oxy^2z^2 + 2Fxy^2z^2 + 2Ixy^3z^2 + 2Ny^4z^2 + Hx^2z^2 + My^3z^2)z' = 0$$

Let $y = kx$ in the equation of the tangent and in the equation
 # See page 4

38

of the curve. Collecting the coefficients of x^3 , x^2z , xz^2 and z^3 in each equation and putting each coefficient in the former equation equal to $-\lambda$ times the corresponding coefficient in the latter we obtain;

$$4Kk y' + (Mk + H)z' + Kk\lambda = 0$$

$$4b^2 x' + (2Kk + 3Mb^2) y' + (2Nb^2 + 2Ik + 2F)z' + (Mb^3 + Kb^2)\lambda = 0$$

$$(2F + Ik)x' + (I + 2Nk)y' + (3Ok + J)z' + (Nb^2 + Ik + F)\lambda = 0$$

$$Jx + 0 y' + (Ok + J)\lambda = 0$$

Eliminating x' , y' , z' and λ from the four linear equations we obtain the determinant:

0	$4Kk$	$Mk + H$	Kk
$4b^2$	$2Kk + 3Mb^2$	$2(Nb^2 + Ik + F)$	$Mb^3 + Kb^2$
$2F + Ik$	$I + 2Nk$	$3(Ok + J)$	$Nb^2 + Ik + F$
J	0	0	$Ok + J$

30

This equation is of the fourth degree in "b". Hence there are four lines of type II which pass through (0, 0, 1).

$$S = 10 \quad R = 4$$

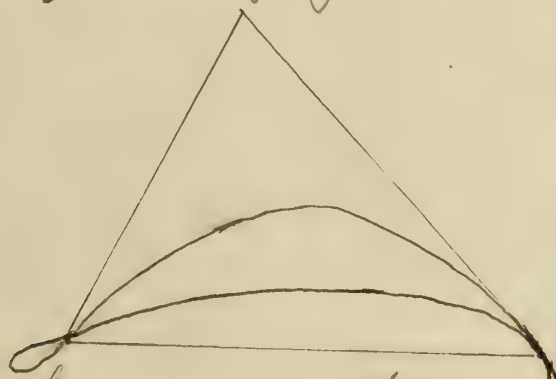
$$S + R = 10 + 4 = 14.$$

Therefore the class of the required locus curve is fourteen.

One Tacnode and One Node

The equation in homogeneous coordinates of the quartic having a tacnode at $(1, 0, 0)$ and a node at the point $(0, 0, 1)$ is;

$$7x^2z^2 + 2xy^2z^2 + 2y^2z^3 + 4xy^2z + 3y^3z + Ky^4 = 0.$$



To determine the number of lines through $(0, 0, 1)$ which join the points of tangency[#] of two tangents which meet on one of the nodal tangents at $(0, 0, 1)$, find the equation of the tangent at (x, y, z) which is;

$$[2(7xz^2 + 2yz^2 + 2y^2z) + (2xz^2 + 2Myz^2 + 2Kxy + 3My^2z + 4Ky^3)]y + [2Fx^2z + 2(2xy^2 + 2My^2z + 4xy^2 + My^3)]z = 0$$

Let $y = kx$ in the equation of the tangent and in the equation of the

See page 5!

Curve. Collecting the coefficients of x^3 , x^2y , x^2z and y^3 in each equation and putting each coefficient in the former equation equal to $-\lambda$ times the coefficient of x^3 , x^2y , x^2z and y^3 in the latter we obtain:

- (1) $4Kk y' + (Mk + H) z' + (Kb^2) \lambda = 0$
- (2) $(Kb^2) x' + (2Kk + 3Mb^2) y' + (2Nk^2 + 2Ik + 2F) z' + (Mb^3 + Hb^2) \lambda = 0$
- (3) $(2F + Ik) x' + (2 + 2Nk) y' + (Nb^2 + Ik + F) \lambda = 0$

Let $I = F + N$ then $x' + y' = 0$ is one of the tangents at the node. Eliminating x' , y' , z' and λ from the above three equations and the equation $x' + y' = 0$ of the tangent we obtain the determinant:

0	4Kk	Mk + H	Kb ²	= 0
Kb ²	2Kk + 3Mb ²	2(Nk ² + Ik + F)	Mb ³ + Hb ²	
2F + Ik	2 + 2N	0	Nb ² + Ik + F	
1	1	0	0	

This equation is of the fourth degree in " k " but as $(k+1)$ is a factor it reduces to one of the third degree.

Hence there are three lines through $(0, 0, 1)$ which are determined by the tangent $x' + y' = 0$

The same number are determined by the second nodal tangent.

The two nodal tangents at $(0, 0, 1)$ are lines of the locus as are also the $n - 4 = 2$ other tangents to the given quartic which can be drawn from $(0, 0, 1)$.

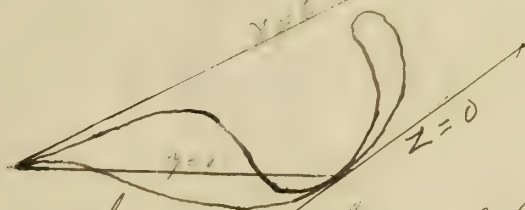
The sum of the number of lines of the three different kinds[#] is:

$$6 + 2 + 2 = 10.$$

Hence the class of the required locus is ten.

One Tacnode and One Cusp.

The equation of the quartic with a tacnode at $(1, 0, 0)$ and a cusp at $(0, 0, 1)$ is:

$$7^2 x^2 z^2 + 2F11 x y z^2 + 11^2 y^2 z^2 + 4 x y^2 z + 11 y^3 + 11 y^4 = 0$$


To determine the number of lines through $(0, 0, 1)$ which join the points of tangency[#] of two tangents which meet on the cuspidal tangent at $(0, 0, 1)$ form the equation of the tangent at (x, y, z) which is.

$$2 \cdot 7^2 x z^2 + 2F11 y z^2 + 4 y^2 z) x' + [2F11 x z^2 + 211^2 y z^2 + 24 x y z + 311 y^2 z + 411 y^3] y' + [2F^2 x^2 z + 4F11 x y z + 211^2 y^2 z + 11 x y^2 + 11 y^3] z' = 0.$$

Let $y = kx$ in the equation of the tangent and in the equation of the curve.

Collecting the coefficients of x^3 , $x^2 z$, $x z^2$ and z^3 in each equation and putting each coefficient in the former equation equal to $\rightarrow$ times the corresponding coefficient

[#] See Page 5

in the latter we obtain:

44

$$(4Kk)y' + (Mk + H)z' + (Kb^2)\lambda = 0$$

$$(Hb^2)x' + (2Hk + 3Mb^2)y' + (2N^2b^2 + 4FNk + 2F^2)z' + (Mb^3 + Kb^2)\lambda = 0$$

$$[2F^2 + 2FNk]x' + [2FN + 2N^2k]y' + [Mb^2 + Hk + F]\lambda = 0$$

Let $F = H$. Then $x' + y' = 0$ is the equation of the tangent at the cusp. Eliminating x', y', z' and λ from the above three equations and the equation $x' + y' = 0$ of the tangent we obtain the determinant.

0	4Kk	Mb + H	Kb ²	
Hb ²	2Hk + 3Mb ²	2(N ² b ² + 2FNk + F ²)	Mb ³ + Kb ²	
2F ² + 2FNk	2FN + 2N ² k	0	Nb ² + 2FNk + F ²	= 0
1.	1.	0	0	

This equation is of the fourth degree in 'b' but as $(k+1)^2$ is a factor it reduces to one of the second degree. Hence

there are two lines through $(0, 0, 1)$ ⁴⁵
but each of these lines must be
counted twice.

The cuspidal tangent at $(0, 0, 1)$ is
a line of the locus. Since $n-3=2$,
there are two other tangents to the given
quartic which can be drawn from $(0, 0, 1)$

The sum of the number of lines of
the three different kinds[#] is

$$4 + 1 + 2 = 7$$

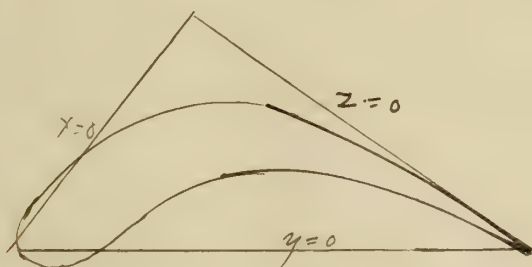
Hence the class of the required
locus is seven.

[#] See Page 5.

The Ramphoid-cusp.

The equation of the quartic with a ramphoid cusp at $(1, 0, 0)$ and passing through $(0, 0, 1)$ is:

$$x^3 + 3Fxy^2 + 3Lx^2z + 3Hxy^2z + 3Mx^2z^2 + 3Nxyz^2 + 3Kx^2z^3 + 3Lxyz^3 + 3Mx^2z^4 = 0. \quad \text{Let } H^2 = 4FK.$$



The number of lines of type I is equal to $(m-2)(n-2) - p - 1^* = 3$
 $(4-2)(7-2) - 1 - 1 = 3$

To determine the number of lines of type II form the equation of the tangent at (x, y, z) which is:

$$(x^3 + 3Fxy^2 + 3Lx^2z + 3Hxy^2z + 3Mx^2z^2 + 3Nxyz^2 + 3Kx^2z^3 + 3Lxyz^3 + 3Mx^2z^4)x' + (3x^2 + 3Fxy + 3Lxz + 3Hxyz + 3Mxz^2 + 3Nyz^2 + 3Kxz^3 + 3Lyz^3 + 3Mxz^4)y' + (3x^2 + 3Fxy + 3Lxz + 3Hxyz + 3Mxz^2 + 3Nyz^2 + 3Kxz^3 + 3Lyz^3 + 3Mxz^4)z' = 0$$

$H^2 = 4FK.$

Let $y = kx$ in the equation of the tangent and in the equation of the curve. Collecting the coefficients of

See page 4.

47

x^3, x^2z, xz^2 and z^3 in each equation, and putting each coefficient in the former equation equal to $-\lambda$ times the corresponding coefficient in the latter we obtain;

$$4Kk y' + (Mk + H)z' + Kb^2 \lambda = 0$$

$$Hb^2 x' + (2Hk + 3Mb^2)y' + (2Nb^2 + 2Sk + 2F)z' + (Mb^3 + Kb^2)\lambda = 0$$

$$(2F + Sk)x' + (I + 2Nk)y' + (3Ok + 3Of)z' + (Nb^2 + Sk + F)\lambda = 0$$

$$Jx + Oy + (Ok + f)\lambda = 0$$

Eliminating x', y', z' and λ from the four linear homogeneous equations we obtain the determinant;

$$\begin{vmatrix} 0 & 4Kk & Mk + H & Kb^2 \\ Hb^2 & 2Hk + 3Mb^2 & 2Nb^2 + Sk + F & Mb^3 + Kb^2 \\ 2F + Sk & I + 2Nk & 3Ok + 3Of & Nb^2 + Sk + F \\ J & 0 & 0 & Ok + f \end{vmatrix} = 0$$

Let $H = 4FK$.

48

In the determinant multiply the first row by $2F$, the second by H , subtract the first line from the second and remember $H^2 = 4FK$ we have after dividing second line by " k " the determinant:

$$\begin{vmatrix}
 0 & +xk & Mb + H & Kb^2 \\
 4FKk & 3Hmk & 2(2H-FM) + 2Hk & MHb^2 + 2FKb \\
 2F + k & 1 + 2mk & 3(Ok + J) & F + k + Mb^2 \\
 J & 0 & 0 & Ok + J
 \end{vmatrix} = 0$$

This equation is of the third degree in " k ". Hence there are three lines of type $\#$. through $(0, 0, 1)$

$$R = 3 \quad S = 8$$

$$R + S = 11.$$

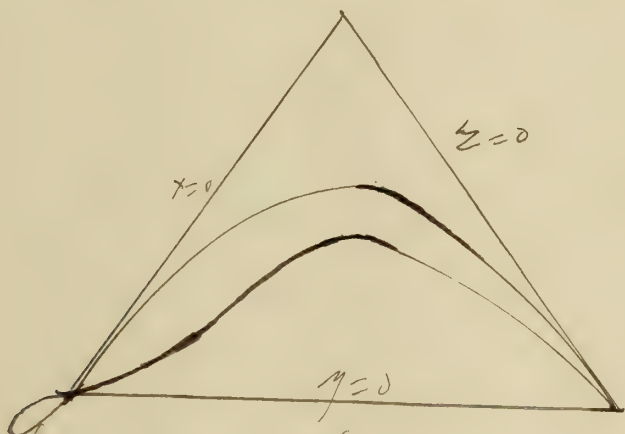
Therefore the class of the required locus curve is eleven!

Ramphoid Cusp and a Node.

The equation of the quartic with a ramphoid cusp at $(1, 0, 0)$ and a node at $(0, 0, 1)$ is;

$$F x^2 z^2 + 2 L x y z^2 + M y^2 z^2 + 4 x y^2 z + M y^3 + K y^4 = 0$$

$$\text{Let } 4 = 4KF$$



To determine the number of lines through $(0, 0, 1)$ which join the points of tangency of two tangents which meet on one of the nodal tangentsth at $(0, 0, 1)$, form the equation of the tangent at (x, y, z) , which is;

$$(2Fxz^2 + 2yz^2 + 4y^2z)x' + (2xz^2 + 2Myz^2 + 2Lxy + 3My^2z + 4Ly^3)y + (3fxz^2 + 2Fx^2z + 2Lxy + 2My^2z + 4Ly^2 + My^3)z = 0$$

Let $y = kz$ in the equation of the tangent and in the equation of the curve. Collecting the coefficients of x^3, x^2z, xz^2, z^3 in each equation. See page 5.

50

and putting each coefficient in the former equation equal to λ times the corresponding coefficient in the latter we obtain;

$$1) 4Kky' + (Mk + H)z' + Kk^2\lambda = 0$$

$$2) 4k^2x' + (2Hk + 3mk^2)y' + (2Nk^2 + 2Ik + 2F)z' + (Mk^3 + Hk^2)\lambda = 0$$

$$3) (2F + Ik)x' + (I + 2Nk)y' + (Nk^2 + Ik + F)\lambda = 0.$$

Let $I = F + N$ then $x' + y' = 0$ is one of the tangents at the node. Multiply the equation (1) by $2F$ and equation (2) by H . Subtract the new equation (1) from the new equation (2) and let this new equation take the place of the second equation. Then eliminate x', y', z' and λ from the above three equations and the equation $x' + y' = 0$ of the tangent we obtain the determinant;

0	4Kk	Mk + H	Kk ²	
4FKK.	3Hmk	2NkH + 2(IH - FM)	MKk ² + 2HFk	= 0
2F + Ik	I + 2Nk	0	F + Ik + Nk ²	
1	1	0	0	

Let $H^2 = 4FK$.

This equation is of the third degree in " b " but as $(b+1)$ is a factor, it reduces to one of the second degree. Hence there are two lines through $(0,0,1)$ which are determined by the tangent $x'+y'=0$

The same number are determined by the second nodal tangent.

The two nodal tangents at $(0,0,1)$ are lines of the locus, as are also the $n-4=1$ other tangent to the given quartic, which can be drawn from $(0,0,1)$.

The sum of the number of lines of the three different kinds[#] is:

$$4 + 2 + 1 = 7.$$

Hence the class of the required locus is seven

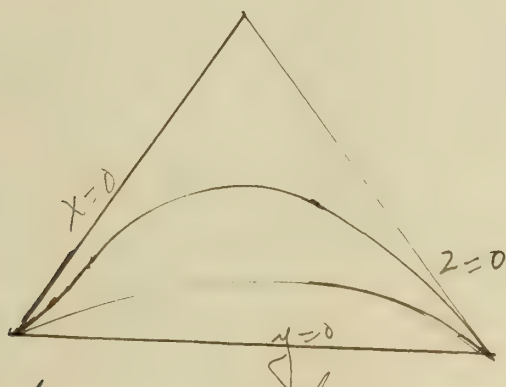
[#] See page 5

Ramphoid-Cusp and a Cusp.

The equation of the quartic with a ramphoid-cusp at $(1, 0, 0)$ and a cusp at $(0, 0, 1)$ is;

$$F^2 x^2 z^2 + 2FHyxyz^2 + H^2 y^2 z^2 + Hxy^2 z + My^3 z + Ky^4 = 0$$

$$\text{Let } H^2 = 4FK.$$



To determine the number of lines through $(0, 0, 1)$ which join the points of tangency of two tangents which meet on the cuspidal tangent at $(0, 0, 1)$, form the equation of the tangent at (x, y, z) which is;

$$(2F^2 x z^2 + 2FHyxyz^2 + Hy^2 z^2)x' + [2FHyxyz^2 + 2H^2 y^2 z^2 + 2Hxy^2 z + 3My^3 z + 4Ky^4]y' + (3F^2 x z^2 + 2F^2 x^2 z + 2Fxyz + 2H^2 y^2 z + 4xy^2 + My^3)z' = 0$$

$$H^2 = 4FK.$$

Let $y = kx$ in the equation of the tangent and in the equation of the curve. Collecting the coefficients

See Page 5.

of x^3, x^2, xz^2 and z^3 in each equation and putting each coefficient in the former equation equal to $-\lambda$ times the corresponding coefficient in the latter we obtain:

$$(1) 4Kky' + (Mk + H)z' + Kb^2\lambda = 0$$

$$(2) Kb^2 + (2Hb + 3Mb^2)y' + (2Nk^2 + 4FNb + 2F^2)z' + (Mb^3 + Hb^2)\lambda = 0$$

$$(3) (2F^2 + 2FNb)x' + (2FN + 2N^2b)y' + (N^2b^2 + 2FNb + F^2)\lambda = 0$$

If $F^2 = N$ then $x' + y' = 0$ is the equation of the tangent at the cusp. Multiply equation (1) by $2F$, equation (2) by H . then subtract the first from the second for a new second equation. Eliminate x', y', z' and λ from the three equations and the equation $x' + y' = 0$ of the tangent we obtain the determinant; Let $H^2 = 4FK$.

0	$4Kb$	$Mb + H$	Kb^2	$= 0$
$4F^2Kb$	$3HMb$	$2N + b + 2(2FNH - F^2M)$	$Mb^2 + KF^2b$	
$2F^2 + 2FNb$	$2FN + 2N^2b$	0	$F^2 + 2FNb + N^2b^2$	
1	1	0	0	

The equation is of the third degree in " h " but as $(h+1)^2$ is a factor, it reduces to one of the first degree. Hence there is one line through $(0, 0, 1)$ which is determined by the tangent $x' + y' = 0$ but this line must be counted twice.

The cuspidal tangent at $(0, 0, 1)$ is a line of the locus. Since $n-3=1$ other tangent to the given quartic curve, which can be drawn from $(0, 0, 1)$

The sum of the number of lines of the three different kinds is:

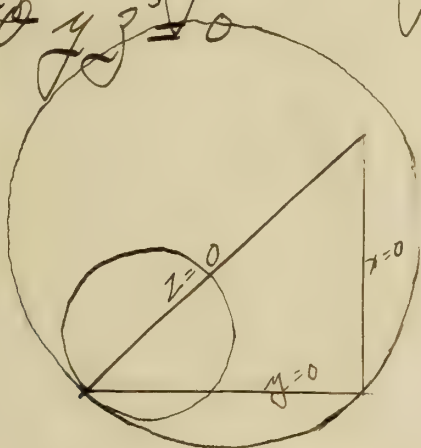
$$2 + 1 + 1 = 4.$$

Hence the class of the required locus is four.

Oscnode.

The equation in homogeneous coordinates of the quartic curve with an oscnode at the point $(1, 0, 0)$ and passing through the point $(0, 0, 1)$ is:

$$x^2 z^2 - H x y^2 z + K^2 y^4 + L x y z^2 - M y^3 z + N x z^3 + O y^2 z^2 + P y z^3 = 0$$



The number of lines of type I is equal to $[m-2][n-2] - p - 2^{\#} = S$
 $[4-2][4-2] - 0 - 2 = 6 \therefore S = 6$

To determine the number of lines of type II, form the equation of the tangent at (x, y, z) which is;

$$[2xz^2 + 2yz^2 - Hzy^2 + z^3]x' + [-2Hxy^2 + 4K^2y^3 + 2xz^2 - 2My^2z + 2Oyz^2 + Oz^3]y' + [2xz^2 - Hxy^2 + 2Lxyz - My^3 + 3Nxz^2 + 2Oyz^2 + 3Pyz^2]z' = 0$$

Let $x=y$ in the equation of the tangent and in the equation of the curve.
[#] see page

Collecting the coefficients of x^3, x^2z, xz^2 and z^3 in each equation and putting each coefficient of x^3, x^2z, xz^2 and z^3 in the former equation equal to $-\lambda$ times the corresponding coefficient in the latter we obtain:

$$4K^2k y' - [14 + Mk]z' + K^2k^2 \lambda = 0$$

$$-4k^2 x' - [24k + 3Mb^2]y' + [2 + 2dk + 2dk^2]z' + [-4k^2 - Mb^3]\lambda = 0$$

$$[2 + dk]x' + [2 + 2dk]y' + [3] + 3dk]z' + [1 + dk + dk^2]\lambda = 0$$

$$x' + 0y' + (1 + dk)\lambda = 0$$

Eliminating x', y', z' and λ from the four homogeneous equations we obtain the following determinant:

$$\begin{vmatrix} 0 & 4K^2k & -4 - Mk & K^2k^2 \\ -4k^2 & -24k - 3Mb^2 & 2(1 + dk + dk^2) & -4k^2 - Mb^3 \\ 2 + dk & 2 + 2dk & 3 + 3dk & 1 + dk + dk^2 \\ 1 & 0 & 0 & 1 + dk \end{vmatrix} = 0$$

Let $M = 24$ and $k = 2K$.

This equation is of third degree in " k ". Hence there are three lines of type Π through $(0, 0, 1)$.

$$S = 6 \quad R = 3$$

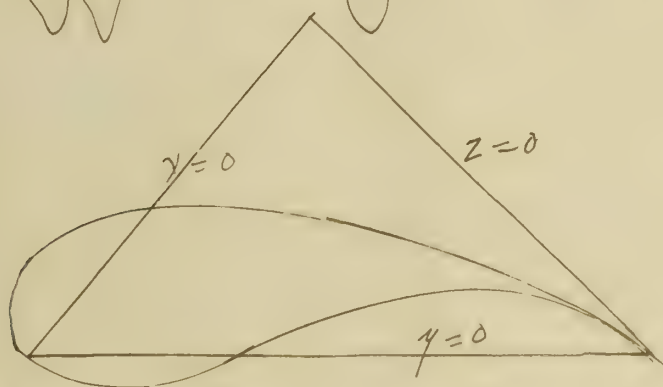
$$R + S = 3 + 6 = 9.$$

Therefore the class of the required locus curve is nine.

Jacnode-Cusp.

The equation of a quartic curve with a Jacnode-cusp at $(0, 1, 1)$, and the curve passing through the point $(0, 0, 1)$ is;

$$x^4[-B+d^2] + x^3[-Az + 2cdz - 2dy] + x^2[y^2 + Cz^2 - 2Czy + 2dmz^2] + x[-2myz^2 + 2cmz^3] = 0$$



The number of lines of type I is equal to $[m-2][n-2] - p = 5$
 $[4-2][5-2] - 0 = 2 \cdot 3 = 6$
 $[4-2][5-2] - 0 = 6 - 2 = 4$

To determine the number of lines of type II, form the equation of the tangent at (x, y, z) which is;

$$[4x^3(-B+d^2) + 3x^2(-Az + 2cdz - 2dy) + 2x(y^2 + cz^2 + 2dmz^2 - 2czy) + (-2myz^2 + 2cmz^3)]x' + [-2dx^3 + 2yx^2 - 2cxz^2 - 2mxy^2]y' + [-Ax^3 + 2cdx^3 + 2cx^3 - 2cx^2y + 4dmx^2z - 4mxyz + 4cmxz^2]z' = 0$$

See Page. 4

Let $y = kx$ in the equation of the tangent and in the equation of the curve. Collecting the coefficients of x^3, x^2z, xz^2 and z^3 in each equation and putting each coefficient in the former equation equal to $-\lambda$ times the corresponding coefficient in the latter, we obtain;

$$[4(-B+d^2)-6dk+2b^2]x' + [-2d+2b]y' + [-A+2cd-2ck]z' + [4(-B+d^2)-2dk+b^2]\lambda = 0$$

$$[-3A+6cd-4ck]x' + [-2c]y' + [2c+4dm-4mk]z' + [-A+2cd-2ck]\lambda = 0$$

$$[2c+4dm-2mk]x' - 2my' + 6cmz' + [c+2dm-2mk]\lambda = 0$$

$$2cmx' + 2cm\lambda = 0$$

$-4B+4d^2$ $-6dk+2b^2$	$-2d+2b$	$-A+2cd$ $-2ck,$	$-B+d^2-$ $2dk+b^2$	
$-3A+6cd-4ck$	$-2c$	$2c+4dm-4mk,$	$-A+2cd-2ck$	
$2c+4dm-2mk,$	$-2m$	$6cm$	$c+2dm-2mk$	
1.	0	0	1.	= 0

This equation is of the third degree in "k". Hence there are three lines of type II through (0, 0, 1)

$$S = 4 \quad R = 3.$$

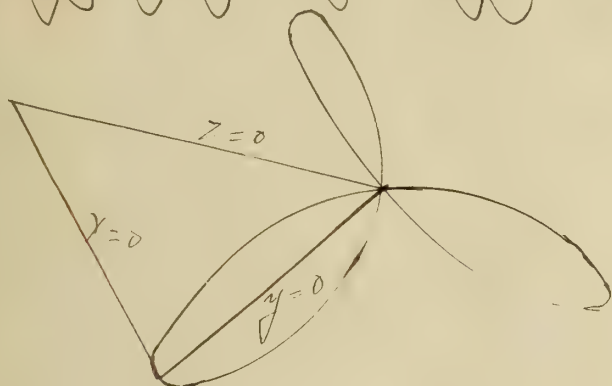
$$S + R = 4 + 3 = 7.$$

Therefore the class of the required locus curve is seven.

Triple Point.

The equation in homogeneous coordinates of the quartic having a triple point at (1, 0, 0) and passing through (0, 0, 1) is:

$$Ax^2y^2 + Bxy^2z + Cx^3 + Ky^4 + My^3z + Ny^2z^2 + Oz^3 = 0$$



The number of lines of type I is equal to $(m-2)(n-2) - p = 5$.

$$[4-2][4-2] - 0 = 4.$$

To determine the number of lines of type II form the tangent equation at (x, y, z) , which is:

$$\begin{aligned} & [4Ay^2z + 2y^2z^2 + z^3]x' + [2Ax^2y + 2x^2z^2 + 4Ky^3 + 3My^2z \\ & + 2Ny^2z^2 + Oz^3]y + [4x^2y + 2x^2z^2 + 3My^3 \\ & + 2Ny^2z^2 + Oz^2y]z' = 0 \end{aligned}$$

Let $kx = y$ in the equation

of the tangent and in the equation of the curve. Collecting the coefficients of x^3 , x^2z , xz^2 and z^3 in each equation and putting each coefficient in the former equal to $-\lambda$ times the corresponding coefficient in the latter we obtain;

$$4Kkx' + (H + Mk)z' + Kb^2\lambda = 0$$

$$Hkx + (2H + 3Mk)y' + (2I + 3Nk)z' + (Hk + Mb^2)\lambda = 0$$

$$Ikx' + (I + 2Nk)y' + (3J + 3Ok)z' + (Ik + Nb^2)\lambda = 0$$

$$Jx' + 0y' + (J + Ok)\lambda = 0$$

Eliminating x' , y' , z' and λ from the four linear homogeneous equations we obtain the following determinant;

0	$4Kk$	$H + Mk$	Kb^2	
Hk	$2H + 3Mk$	$2I + 3Nk$	$Hk + Mb^2$	$= 0$
Ik	$I + 2Nk$	$3J + 3Ok$	$Ik + Nb^2$	
J	0	0	$J + Ok$	

This equation is of the second degree in "k". Hence there are two lines of type II through (0, 0, 1).

$$S=4 \quad R=3$$

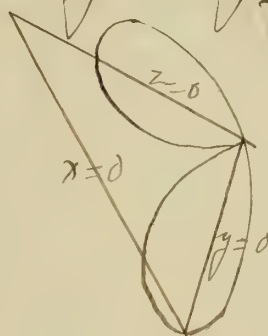
$$S+R=4+3=6$$

Therefore the class of the required locus curve is six.

Triple Point

The equation of the quartic in homogeneous coordinates having a triple point with two tangents coincident and passing through the point $(0, 0, 1)$ is;

$$Lxyz^2 + Lxz^3 + Ky^4 + My^3z + Ny^2z^2 + Oz^3 = 0$$



The number of lines of type I is equal to $(m-2)(n-2) - p = 3$.

$$(4-2)(4-2) - 0 = 4$$

To determine the number of lines of type II, form the tangent equation at (x', y', z') which is;

$$\begin{aligned} & (Lyz^2 + Lz^3)x' + (2Lxz^2 + 4Ky^3 + 3My^2z \\ & + 2Ny^2z + Oz^3)y' + (2Lxy^2 + 2Lxz^2 + My^3 \\ & + 2Ny^2z + 3Oyz^2)z' = 0 \end{aligned}$$

Let $y = kx$ in the equation of the

45

tangent and in the equation of the curve. Collecting the coefficients of x^3 , x^2z , xz^2 and z^3 in each equation and putting each coefficient in the former equals to $-\lambda$ times the corresponding coefficient in the latter we obtain:

$$4K y' + M z' + K k \lambda = 0$$

$$3 M k y' + (2L + 2 N k) z' + M k^2 \lambda = 0$$

$$L k x' + (L + 2 N k) y' + (3 J + 3 O k) z' + (L k + N k^2) \lambda = 0$$

$$J x' + O y' + (J + O k) \lambda = 0$$

Eliminating x' , y' , z' and λ from these four linear homogeneous equations we obtain the following determinant:

0	4K	~~M~~	Kk	= 0
0	3mk	2L + 2Nk	Mk ²	
Lk	L + 2Nk	3J + 3Ok	Lk + Nk ²	
J	O	0	J + Ok	

This equation is of the second ⁶⁶ degree in " k ". Hence there are two lines of type II through $(0, 0, 1)$

$$S = 4 \quad R = 2.$$

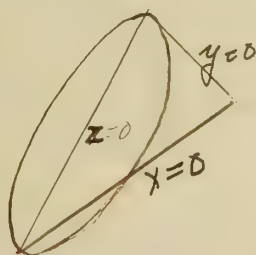
$$S + R = 4 + 2 = 6.$$

Therefore the class of the required locus curve is six.

A Triple Point.

The equation in homogeneous co-ordinates of the quartic with a triple point, whose three tangents are coincident, and which passes through $(0, 0, 1)$ is:

$$Jz^3 + Ky^4 + My^3z + Ny^2z^2 + Oz^3 = 0$$



The number of lines of type I is equal to $[m-2][n-2]-p$

$$[4-2][4-2]-0 = 4 \quad \therefore S = 4.$$

To determine the number of lines of type II, form the equation of the tangent at (x, y, z) which is;

$$[Jz^3]x' + [4Ky^3 + 3My^2z + 2Ny^2z^2 + Oz^3]y' + [Jx + My^3 + 2Ny^2z + 3Oyz^2]z' = 0$$

Let $y = kx$ in the equation of the tangent and in the equation of

the curve. Collecting the coefficients of x^3 , x^2z , xz^2 and z^3 in each equation and putting each coefficient in the former equal to $-\lambda$ times the coefficient of the latter, we obtain;

$$4K y' + M z' + K k \lambda = 0$$

$$3M y' + 2N z' + M k \lambda = 0$$

$$2N k y' + 3J z' + 3O k z' + M k^2 \lambda = 0$$

$$J x' + O y' + [J + O k] \lambda = 0$$

Eliminating x' , y' , z' and λ from these four linear homogeneous equations we obtain the following determinant:

0	4K	M	Kk	
0	3M	2N	Mk	= 0
0	2Nk	3J + 3Ok	Mk ²	
J	O	0	J + Ok	

43

This equation is of the first degree in "k". Hence there is one line of type II through (0, 0, 1)

$$S = 4 \quad R = 1$$

$$S + R = 4 + 1 = 5.$$

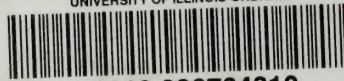
Therefore the class of the required locus curve is five.

Curve	Class of locus.
No Singularities	24.
One Double point	20.
One Cusp	17
Two double points	16
One double point and one Cusp.	13.
Two Cusps	10.
Three nodes	12
Two nodes and one Cusp	9
One node and two cusps	6
Three cusps	3
Tacnode	14
Tacnode and one node	10
Tacnode and one cusp	7
Ramphoid cusp	11
Ramphoid cusp and node	7
Ramphoid cusp and cusp	4.
Osnode	9
Tacnode-cusp	7
Triple points	
Tangents distinct	40
Two tangents coincident	6
Three tangents coincident	5





UNIVERSITY OF ILLINOIS-URBANA



3 0112 086764310